CONCEPT AND APPLICATION OF STOCHASTIC DOMINANCE IN INVESTMENTS

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Abstract: **Stochastic dominance (SD) is a fundamental concept in decision theory. The term is used in decision theory and decision analysis to refer to situations where one gamble (a probability distribution over possible outcomes, also known as prospects) can be ranked as superior to another gamble. It is based on preferences regarding outcomes. It is associated with choice, on outcome of distribution and uncertainty in investment parlance. It is a form of stochastic ordering. A preference might be a simple ranking of outcomes from favorite to least favored, or it might also employ a value measure (i.e., a number associated with each outcome that allows comparison of multiples of one outcome with another, such as two instances of winning a dollar vs. one instance of winning two dollars.)**

Keywords: **Stochastic dominance (SD), investment, Decision theory.**

1. INTRODUCTION

Stochastic dominance is a method of comparing random selection of variables in investment portfolio which reveals the personality, perception and preference on particular portfolio performance over the other, regardless of the nature and type of the investor.

Stochastic dominance encompasses expectation, utility, future wealth on and of respective portfolio desire whose performance and distribution is superior to another alternative i.e. based on utility and returns.

Stochastic dominance describes when a particular randomly selected prospect A is better than another randomly selected prospect B in a portfolio based on their performance outcome and the preference of the investor.

In essence of the investor performance outcome of $A \leq B$, we say B stochastically Dominated A. This notions as explained above means that we need to know about

   i. Distribution wXi for each i=1, 2, 3……………..n.

   ii. And utility function u(x).

It is the knowledge or assumption of these two that allows us to say one variable perform better than the other. This is stochastic dominance,

Where $F X(x) = P(X \leq x)$.

2. TYPES

Consider two portfolios A and B with utility (u) and return (r) associated with each.

Where $P (A \leq B) = 1$ in either u or r or both.

Then random selection of B will perform better than A because the probability is already given as 1 (i.e. assured or certain). This means it is sure that B will outperform A. This is state by state dominance. The issue in reality is how certain is certain- Probability/uncertainty.

Probability therefore becomes a major issue in this consideration i.e. uncertainty as it affects performance and distribution.

There are several types of stochastic dominance.

a) Absolute Dominance - A.D= Almost sure.

b) State dominance

c) First Order Stochastic Dominance -FSD=more common where whatever value of u is wealthy increasing.

d) Second Order Stochastic Dominance -SSD where an investor is risk averse but it is still weakly increasing.

e) Third order stochastic dominance

f) Higher order stochastic dominance

But they are all related i.e. AD=>FSD=>SSD. =>TSD

However they have descending order of strength.

> *NOTE*: - FSD and SSD are properties of distribution concerned i.e. A and B. while AD is concerned with a particular or actual random variable.

Time frame is asserted in SD essentially for FSD and SSD i.e period of investment and returns and not necessarily the amount of investment or amount invested.(capital)

State Wise Dominance

The simplest case of stochastic dominance is **statewise dominance** (also known as

state-by-state dominance), defined as follows: gamble A is statewise dominant over gamble B if A gives a better outcome than B in every possible future state (more precisely, at least as good an outcome in every state, with strict inequality in at least one state). For example, if a dollar is added to one or more prizes in a lottery, the new lottery statewise dominates the old one. Similarly, if a risk insurance policy has a lower premium and a better coverage than another policy, then with or without damage, the outcome is better. Anyone who prefers more to less (in the standard terminology, anyone who has monotonically increasing preferences) will always prefer a statewise dominant gamble.

First-order stochastic dominance

Statewise dominance is a special case of the canonical **first-order stochastic dominance**, defined as follows: Gamble A has first-order stochastic dominance over gamble B if for any good outcome x, A gives at least as high a probability of receiving at least x as does B, and for some x, A gives a higher probability of receiving at least x. In notation form, $P[A \geq x] \geq P[B \geq x]$ for all x, and for some x, $P[A \geq x] > P[B \geq x]$. In terms of the cumulative distribution functions of the two gambles, A dominating B means that $F_A(x) \leq F_B(x)$ for all x, with strict inequality at some x. For example, consider a die-toss where 1 through 3 wins \$1 and 4 through 6 wins \$2 in gamble B. This is dominated by a gamble C that yields \$3 for 1 through 3 and \$1 for 4 through 6, and it is also dominated by a gamble A that gives \$1 for 1 and 2 and \$2 for 3 through 6. Gamble A has statewise dominance over B, but gamble C has first-order stochastic dominance over B without statewise dominance. This is because, in states 4 to 6, gamble C has a worse outcome than B, however $P[C \geq x] = P[B \geq x]$ for all $x \leq 2$ and $P[C \geq x] > P[B \geq x]$ for all $2 < x \leq 3$. Further, although when A dominates B, the expected value of the payoff under A will be greater than the expected value of the payoff under B, this is not a sufficient condition for dominance, and so one cannot order lotteries with regard to stochastic dominance simply by comparing the means of their probability distributions.

Every expected utility maximizer with an increasing utility function will prefer gamble A over gamble B if A first-order stochastically dominates B.

First-order stochastic dominance can also be expressed as follows: If and only if A first-order stochastically dominates B, there exists some gamble y such that $x_B \overset{d}{=} (x_A + y)$ where $y \leq 0$ in all possible states (and strictly negative in at least one state); here $\overset{d}{=}$ means "is equal in distribution to" (that is, "has the same distribution as"). Thus, we can go from the graphed density function of A to that of B by, roughly speaking, pushing some of the probability mass to the left.

Second-Order Stochastic Dominance

The other commonly used type of stochastic dominance is **second-order stochastic dominance**. Roughly speaking, for two gambles A and B, gamble A has second-order stochastic dominance over gamble B if the former is more predictable (i.e. involves less risk) and has at least as high a mean. All risk-averse expected-utility maximizers (that is, those with increasing and concave utility functions) prefer a second-order stochastically dominant gamble to a dominated gamble. Thus, For any lotteries F and G, F second-order stochastically dominates G if and only if the decision maker weakly prefers F to G under every weakly increasing concave utility function u.

In terms of cumulative distribution functions F_A and F_B, A is second-order stochastically dominant over B if and only if the area under F_A from minus infinity to x is less than or equal to that under F_B from minus infinity to x for all real numbers x, with strict inequality at some x; that is, $\int_{-\infty}^{x} [F_B(t) - F_A(t)] dt \geq 0$ for all x, with strict inequality at some x. Equivalently, A dominates B in the second order if and only if $E[u(A)] \geq E[u(B)]$ for all non-decreasing and concave utility functions $u(x)$.

Second-order stochastic dominance can also be expressed as follows: If and only if A second-order stochastically dominates B, there exist some gambles y and z such that $x_B \overset{d}{=} (x_A + y + z)$, with y always less than or equal to zero, and with $E(z \mid x_A + y) = 0$ for all values of $x_A + y$. Here the introduction of random variable y makes B first-order stochastically dominated by A (making B disliked by those with an increasing utility function), and the introduction of random variable z introduces a mean-preserving spread in B which is disliked by those with concave utility. Note that if A and B have the same mean (so that the random variable y degenerates to the fixed number 0), then B is a mean-preserving spread of A.

Sufficient conditions for second-order stochastic dominance

- First-order stochastic dominance of *A* over *B* is a sufficient condition for second-order dominance of *A* over *B*.

- If *B* is a mean-preserving spread of *A*, then *A* second-order stochastically dominates *B*.

Necessary conditions for second-order stochastic dominance

- $E_A(x) \geq E_B(x)$ is a necessary condition for *A* to second-order stochastically dominate *B*.

- If A dominates B in the second order, then the geometric mean of A must be greater than or equal to the geometric mean of B.

- $\min_A(x) \geq \min_B(x)$ is a necessary condition. The condition implies that the left tail of F_B must be thicker than the left tail of F_A.

Third-Order Stochastic Dominance

Let F_A and F_B be the cumulative distribution functions of two distinct investments A and B. A dominates B in **the third order** if and only if

- $\int_{-\infty}^{x} \int_{-\infty}^{z} [F_B(t) - F_A(t)] \, dt \, dz \geq 0$ for all x,

- $E_A(x) \geq E_B(x),$ and there is at least one strict inequality. Equivalently, A dominates B in the third order if and only if $E_A U(x) \geq E_B U(x)$ for all non-decreasing, concave utility functions U that are **positively skewed** (that is, have a positive third derivative throughout).

Sufficient condition for third-order stochastic dominance

- Second-order stochastic dominance is a sufficient condition.

Necessary conditions for third-order stochastic dominance

- $E_A(\log(x)) \geq E_B(\log(x))$ is a necessary condition. The condition implies that the geometric mean of A must be greater than or equal to the geometric mean of B.

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- $\min_A(x) \geq \min_B(x)$ is a necessary condition. The condition implies that the left tail of F_B must be thicker than the left tail of F_A.

Higher-order stochastic dominance

Higher orders of stochastic dominance have also been analyzed, as have generalizations of the dual relationship between stochastic dominance orderings and classes of preference functions.

3. IMPORTANCE OR USEFULLNESS

According to Jenkins and Lambert(1997, 1998), Shorocks(1998) and Spencer and Fisher(1992) the whole theory of stochastic dominance can be developed using quantile rather than income approach. This is called p-approach.

Quantile function is defined as the inverse of Cumulative Distribution Function(CDF). This is why generalized Lorenz dominance and Stochastic dominance are equivalent conditions. i.e. there is a linkage within Lorenzic dominance and Stochastic dominance it can be presented graphically and more easily understood.

More so, stochastic dominance is principally concerned with comparing the performance of investment (variables) randomly selected in portfolio management or analysis, based on individual variables or comparing selected random variables.

As previously noted, it is a function of return, utility and period upon which selection decision is made in order to maximize the total wealth of the investor.

Decisions under stochastic dominance is therefore not based on the amount of capital invested (wealth) itself but future prospect based on performance of the variables concerned.

APPLICATION

Generally, everybody expects returns on investment and this desire affects choice on alternative investment as supported by stochastic dominance theory.

Consider an investment opportunity on a particular security S in portfolio P. and the return per unit of investment is S_i (non -negative probability).

Total wealth at the end of the period is wS_i (i.e. returns plus capital invested),

Where i=1, 2, 3…………..n.

Then invest or chose or decide on:

E(u(wSn))= max E (u (wS_i))

$1 \leq i < n$

i.e cumulative value of S at the end of the period is better than at the beginning.

Also choosing between two investment P and S where in particular period t.

$$wPt \leq wSt.$$

It means that S has a better performance than P therefore S will be preferred above P by a rational investor. This is stochastic dominance.

In the application of Portfolio Theory, Second order stochastic dominance plays a role when one begins to construct a framework to analyze risk-adjusted returns. MPT (Modern Portfolio Theory) employs the CAL (Capital Allocation Line) to evaluate the expected return (Mean) on the y-axis and the standard deviation (the square root of variance) x-axis. The whole concept of risk aversion is based on second order stochastic dominance in portfolio selection; however, some of the shortfalls of MPT are a result of not evaluating third order dominance or fourth order dominance criteria i.e. the skewness or kurtosis of the distribution.

"Markowitz (1959) recognized the asymmetrical inefficiencies inherited in the traditional mean-variance approach and suggested a semi-variance measure of asset risk that focuses only on the risks below a target rate or return. Post Modern Portfolio Theory (PMPT) employs the use of Mean Lower Partial Moments as a framework for analyzing risk. "Bawa (1975) generalized the semi-variance measure of risk to reflect a less restrictive class of decreasing absolute risk-averse utility function and shows that the second order mean-LPM for a class of DARA utility functions, is a preferred approximation for the optimal third order stochastic dominance selection rule compared to the mean-variance criteria." Sing and Ong

Stochastic dominance is used in mathematical optimization, in particular stochastic programming. In a problem of maximizing a real functional $f(X)$ over random variables X in a set X_0 we may additionally require that X stochastically dominates a fixed random *benchmark* B. In these problems, utility functions play the role of Lagrange multipliers associated with stochastic dominance constraints. Under appropriate conditions, the solution of the problem is also a (possibly local) solution of the problem to maximize $f(X) + E[u(X) - u(B)]$ over X in X_0, where $u(x)$ is a certain utility function. If the first order stochastic dominance constraint is employed, the utility function $u(x)$ is non-decreasing; if the second order stochastic dominance constraint is used, $u(x)$ is non-decreasing and concave.

Stochastic dominance is also used in risk measurement, applied in actuary forecast, and it one of the tools used in decision making.

Limitation:

Performance and utility are the main issues in SD as regards investment decisions. Nature and personality of the investor are not factored in, it is about rational thinking.

Also amount of wealth or investment is not considered, it is individual unit variable performance (that might affect the entire portfolio). It is a choice of one period return being better than the other, or one randomly selected variables (security/asset) performing better than other in a given portfolio or comparing the performance of one portfolio to the other; not the amount involved or invested rather than the value of the entire portfolio or a single asset at a particular time or given period of time upon which forecast and investment future decision could be based.

 In S.D investors are all deemed to be rational thinkers i.e prefers more to less return, this connote that choice is purely based on quantitative factors and not qualitative reasoning in SD i.e. SD is not human behaviorly oriented.

In FSD, A is preferred over B regardless of the utility of B as long as B performance and value is weakly increasing.

In SSD, A is preferred over B where the investor is risk averse and B is weakly increasing.

4. CONCLUSION

SD is a tool in considering investment alternatives and portfolios performance based on returns, utility and expectation and wealth distribution. There is this argument among the practitioners and the academics on SD such that P$\leq$ S. Practitioners ' view it that S dominates A but the Academics think otherwise. Despite proof upon proofs it is generally

accepted that in reality practitioners are right even in SSD. This means many issues under SD are mere academic exercise when reality and human behavior (qualitative) factors are brought into play as far as returns, expectations, utility and accumulated value or wealth is considered in as much as

$$A \leq S,$$

Then the performance of S is better and it therefore dominates A in SD techniques.

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